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Arithmetic progressions in finite groups

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In this paper, we describe the structure of finite groups whose element orders or proper (abelian) subgroup orders form an arithmetic progression of ratio $r \geq 2$. This extends the case $r = 1$ studied in previous papers [R. Brandl and W. Shi, Finite groups whose element orders are consecutive integers, *J. Algebra* **143** (1991) 388–400; Y. Feng, Finite groups whose abelian subgroup orders are consecutive integers, *J. Math. Res. Exp.* **18** (1998) 503–506; W. Shi, Finite groups whose proper subgroup orders are consecutive integers, *J. Math. Res. Exp.* **14** (1994) 165–166].

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1. Introduction

Given a finite group G, we denote by $\pi_e(G)$ the set of element orders of G, by $\pi_s(G)$ the set of proper subgroup orders of G and by $\pi_{as}(G)$ the set of proper abelian subgroup orders of G. The starting point for our discussion is given by [1, 5, 9] that classify finite groups G in which the elements of the above sets are consecutive integers, i.e. arithmetic progressions of ratio $r = 1$.

This paper deals with the case $r \geq 2$. Then we may assume that G is of odd order, by Cauchy's theorem. Our main results are as follows.

Theorem 1.1. *Let G be a finite group of odd order. Then $\pi_e(G)$ is an arithmetic progression if and only if G is either a p-group of exponent p or a Frobenius group with kernel $P \in \mathrm{Syl}_p(G)$ of exponent p and complement $Q \in \mathrm{Syl}_q(G)$ of order q, and moreover $p = 2q - 1$ or $q = 2p - 1$.*

It is not known whether there are infinitely many pairs of primes (q, p) with $p = 2q - 1$.[a] So, we cannot decide whether there are infinitely many groups of the above second kind. Note that the smallest one is of order 75, namely SmallGroup$(75, 2) = [P]Q$ from GAP [11], where $P \cong C_5 \times C_5$ and $Q \cong C_3$.

Theorem 1.2. *Let G be a finite group of odd order. Then $\pi_s(G)$ or $\pi_{as}(G)$ is an arithmetic progression if and only if either $G \cong C_p \times C_p$ or G is cyclic of one of the following types: C_{p^i}, $i = 1, 2$, or C_{pq} with p and q distinct primes such that $p = 2q - 1$ or $q = 2p - 1$.*

Together with the results of [1, 5, 9], these lead to a complete classification of finite groups G for which $\pi_e(G)$, $\pi_s(G)$ or $\pi_{as}(G)$ are arithmetic progressions.

For the proof of Theorems 1.1 and 1.2, we need the classification of finite groups with all nontrivial elements of prime order and the classification of finite minimal non-cyclic groups.

Theorem 1.3 ([3, 4]). *Let G be a finite groups having all nontrivial elements of prime order. Then*

(a) *G is nilpotent if and only if G is a p-group of exponent p.*
(b) *G is solvable and non-nilpotent if and only if G is a Frobenius group with kernel $P \in \mathrm{Syl}_p(G)$, with P a p-group of exponent p and complement $Q \in \mathrm{Syl}_q(G)$, with $|Q| = q$. Moreover, if $|G| = p^n q$ then G has a chief series $G = G_0 > P = G_1 > G_2 > \cdots > G_k > G_{k+1} = 1$ such that for every $1 \leq i \leq k$ one has $G_i/G_{i+1} \leq Z(G/G_{i+1})$, Q acts irreducibly on G_i/G_{i+1} and $|G_i/G_{i+1}| = p^b$, where b is the exponent of $p \pmod q$.*
(c) *G is non-solvable if and only if $G \cong A_5$.*

Theorem 1.4 ([8]). *A finite group G is a minimal non-cyclic group if and only if it is isomorphic to one of the following groups:*

(a) *$C_p \times C_p$, where p is a prime;*
(b) *Q_8;*
(c) *$\langle a, b \,|\, a^p = b^{q^m} = 1, b^{-1}ab = a^r \rangle$, where p and q are distinct primes and $r \not\equiv 1 \pmod p$, $r^q \equiv 1 \pmod p$.*

We also need the following result.

Lemma 1.5 (Lucidos three primes lemma [7]). *Let G be a finite solvable group. If p, q, r are distinct primes dividing $|G|$, then G contains an element of order the product of two of these three primes.*

Finally, we formulate an open problem related to Theorem 1.2.

[a]These pairs are known as *Cunningham chains of the second kind of length* 2 — see e.g. the sequence A005382 in [10].

Open problem. Determine all finite groups for which the set of proper normal subgroup orders is an arithmetic progression.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [6].

2. Proofs of the Main Results

We start with the following lemma.

Lemma 2.1. *Let G be a finite group of odd order.*

(a) *If $\pi_e(G)$ is an arithmetic progression, then all nontrivial elements of G are of prime order.*
(b) *If $\pi_s(G)$ or $\pi_{as}(G)$ is an arithmetic progression, then all non-trivial proper subgroups of G are of prime order.*

Proof. (a) Let $\pi_e(G) = \{a_0 = 1, a_1, \ldots, a_n\}$. Then a_1 is the smallest prime p dividing $|G|$ and so $a_i = ip - i + 1$, for all i. In particular, we have $a_{p+1} = p^2$. Since for every $d \in \pi_e(G)$ the set of divisors of d is contained in $\pi_e(G)$, we infer that $a_2, a_3, \ldots, a_p$ are also primes. We distinguish the following two cases:

Case 1. $p = 3$.

Then $\pi_e(G) = \{1, 3, 5, 7, \ldots, m\}$.

Assume first that $m \geq 41$. Then $[\frac{m}{8}] \geq 2$, and thus there are three primes p_1, p_2 and p_3 such that

$$\left[\frac{m}{8}\right] < p_1 < 2\left[\frac{m}{8}\right] < p_2 < 4\left[\frac{m}{8}\right] < p_3 < 8\left[\frac{m}{8}\right] \leq m,$$

by the well-known Bertrand's postulate. Since G is solvable, Lemma 1.5 shows that there exists $a \in G$ with $o(a) \in \{p_1p_2, p_2p_3, p_3p_1\}$. This leads to a contradiction since

$$o(a) \geq p_1p_2 > 2\left[\frac{m}{8}\right]^2 > m,$$

where the last inequality follows from our assumption.

Assume now that $m \in \{7, 9, \ldots, 39\}$. If $m \leq 13$, then $3, 5, 7 \in \pi_e(G)$ and so there exists $a \in G$ with $o(a) \in \{15, 21, 35\}$ by Lemma 1.5; thus $o(a) > 13$, a contradiction. If $m > 13$, then $7, 11, 13 \in \pi_e(G)$ and similarly we get $a \in G$ with $o(a) \in \{77, 91, 143\}$, that is $o(a) > 39$, a contradiction.

Consequently, $m \leq 5$, i.e. $\pi_e(G) \subseteq \{1, 3, 5\}$, showing that all non-trivial elements of G are of prime order.

Case 2. $p \geq 5$.

Let q be the smallest prime not dividing $p - 1$. Then $q < p$. On the other hand, in the arithmetic progression $\pi_e(G)$ every qth term is divisible by q (see e.g., [2, Exercise 18(a), p. 58]). Thus $q \mid a_i$ for some $i = 1, 2, \ldots, q - 1$, implying that $a_i = q$

because a_i is prime. Then $q = a_i \geq a_1 = p$, a contradiction. This completes the proof.

(b) Suppose that $\pi_s(G)$ or $\pi_{as}(G)$ is an arithmetic progression, say $\{a_0 = 1, a_1, \ldots, a_n\}$. Again, we infer that a_1 is the smallest prime p dividing $|G|$, $a_i = ip - i + 1$, for all i, and that $a_2, a_3, \ldots, a_p$ are also primes. Then the proof follows similarly to the proof of a). We only note that, given any three distinct primes dividing $|G|$, there is a (abelian) subgroup of G of order the product of two of these three primes by Lemma 1.5. □

We are now able to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let G be a finite group of odd order such that $\pi_e(G)$ is an arithmetic progression. Then all nontrivial elements of G are of prime order by Lemma 2.1(a), and therefore G is either a p-group of exponent p or a Frobenius group with kernel $P \in \mathrm{Syl}_p(G)$ of exponent p and complement $Q \in \mathrm{Syl}_q(G)$ of order q by Theorem 1.3. Clearly, in the second case, we have $\pi_e(G) = \{1, p, q\}$, implying that $p = 2q - 1$ or $q = 2p - 1$, as desired. □

Proof of Theorem 1.2. Let G be a finite group of odd order such that $\pi_s(G)$ or $\pi_{as}(G)$ is an arithmetic progression. Then all nontrivial proper subgroups of G are cyclic by Lemma 2.1(b) It follows that G is either cyclic or minimal non-cyclic. In the first case, we get $G \cong C_{p^i}$, $i = 1, 2$, or $G \cong C_{pq}$ with p and q distinct primes such that $p = 2q - 1$ or $q = 2p - 1$, while in the second one we get $G \cong C_p \times C_p$ by Theorem 1.4 (note that we cannot have $G \cong \langle a, b \mid a^p = b^{q^m} = 1, b^{-1}ab = a^r \rangle$ because Lemma 2.1(b), would imply $m = 1$, i.e. G would be a non-abelian group of order pq, and so $\pi_s(G) = \pi_{as}(G) = \{1, p, q\}$; assuming $p < q$, we find $p \mid q - 1$ and $q = 2p - 1$, leading to $p \mid 2p - 2$, a contradiction). This completes the proof. □

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